Lecture 4: Focusing

What shall we do when we run out of possibilities when we do inversion? The interesting observation is that we may pick one assumption to work on and apply non-invertible rules as eagerly as possible, into so called chains. This might be surprising. It turns out, that you never have to backtrack within one of those chains. Either you need the entire chain to complete, or you don’t need to work on the chosen assumption at all. This is one of the insights that is due to Andreoli [And92]. To make this idea precise, we introduce a focus $[A]$. In our hypothetical judgment $\Gamma; \Delta \implies A$ we may have at most one focus. No focus means that we are still inverting, one focus simple singles our that we are in the middle of chain apply non-invertible rules. Not to confuse things, we write $\Gamma; A \rightarrow \gamma$ for this judgment, where we define

$$\begin{align*}
\delta & ::= \cdot | \delta, A | \delta, [A] \\
\gamma & ::= A | [A]
\end{align*}$$

First, we only consider the fragment without persistent resources. We keep the $\Gamma;$, but we will consider it later.

$$\begin{align*}
\Gamma; [P^-] & \rightarrow P^- \quad \Gamma; P^+ \rightarrow [P^+] \\
\Gamma; (\delta, A) & \rightarrow B \quad \Gamma; \Delta_1 \rightarrow [A] \quad \Gamma; \Delta_2, [B] \rightarrow C \\
\Gamma; \delta & \rightarrow A \rightarrow B \quad \Gamma; (\Delta_1, \Delta_2, [A \rightarrow B]) \rightarrow C \quad \rightarrow L \\
\Gamma; \Delta_1 & \rightarrow [A] \quad \Gamma; \Delta_2 \rightarrow [B] \\
\Gamma; \Delta & \rightarrow [A \otimes B] \\
\Gamma; \Theta & \rightarrow 1 \\
\Gamma; \delta & \rightarrow A[a/x] \\
\Gamma; \delta & \rightarrow \forall x: \tau. A
\end{align*}$$

Next, we consider how to enter a chain an how to leave it. There are two rules for entering,

$$\begin{align*}
\Gamma; \Delta & \rightarrow [A^+] \quad \text{focusR} \\
\Gamma; \Delta & \rightarrow A^+ \\
\Gamma; \Delta & \rightarrow [A^-] \quad \text{focusL} \\
\Gamma; \Delta & \rightarrow A^- \\
\Gamma; \Delta & \rightarrow [A^-]
\end{align*}$$

and two rules exiting (called blurring), if you read the rules bottom up.

$$\begin{align*}
\Gamma; \Delta & \rightarrow A^- \quad \text{blurR} \\
\Gamma; \Delta & \rightarrow [A^-] \\
\Gamma; \Delta & \rightarrow A^+ \quad \text{focusR} \\
\Gamma; \Delta & \rightarrow [A^+] \\
\Gamma; \Delta & \rightarrow C
\end{align*}$$
In the last lecture we proved the initiality extension and the admissibility of the cut rule for our logic. By permitting to focus on assumptions, we need to generalize both induction hypothesis of the admissibility theorem, by the following admissible rules. Here the admissible rules initiality expansion

\[
\begin{array}{c}
\Gamma; A \to A \\
\Gamma; A \to [A] \\
\Gamma; [A] \to A
\end{array}
\]

Next, the admissible cut rules.

\[
\begin{array}{c}
\Gamma; \Delta \to [A] & \Gamma; (\delta, A) \to C & \frac{}{\Gamma; (\Delta, \delta) \to C} \text{cut}_L \\
\Gamma; \delta \to A & \Gamma; (\delta, [A]) \to C & \frac{}{\Gamma; (\Delta, \delta) \to C} \text{cut}_R \\
\Gamma; \Delta \to A^- & \Gamma; (\delta, A^-) \to C & \frac{}{\Gamma; (\Delta, \delta) \to C} \text{cut}_L \\
\Gamma; \delta \to A^+ & \Gamma; (\delta, A^+) \to C & \frac{}{\Gamma; (\Delta, \delta) \to C} \text{cut}_R
\end{array}
\]

The main theorem is due to Andreoli, shows that focusing provability. We need to show two directions.

**Theorem 16 (Soundness)** Let \( \Gamma; \delta \to \gamma \). If we erase all focusing brackets from \( \delta \) and \( \gamma \), we obtain \( \Delta \) and a formula \( C \). Then \( \Gamma; \Delta \Rightarrow C \).

**Proof:** Easy induction. Remove focusing and blurring rules.

**Theorem 17 (Completeness)** If \( \Gamma; \Delta \Rightarrow C \) then \( \Gamma; \Delta \to C \).

**Proof:** This proof is much more complicated. We show only one case that \( D \) ends in the \( \to L \) rule.

\[
\begin{array}{c}
D_1 \quad D_2 \\
\Gamma; \Delta_1 \Rightarrow A & \Gamma; (\Delta_2, B) \Rightarrow C & \frac{}{\Gamma; (\Delta_1, \Delta_2, A \to B) \Rightarrow C} \to L
\end{array}
\]

By applying the induction hypothesis on \( D_1 \) and \( D_2 \), we obtain that \( \Gamma; \Delta_1 \to A \) and \( \Gamma; (\Delta_2, B) \to C \). We need to show that \( \Gamma; (\Delta_1, \Delta_2, A \to B) \to C \). Let’s start bottom up, and apply the focusing rule \( \text{focus}_L \).

\[
\Gamma; (\Delta_1, \Delta_2, [A \to B]) \to C
\]

This can only be achieved by applying \( \to \), now we only need to show \( \Gamma; \Delta_1 \to [A] \) and \( \Gamma; (\Delta_2, [B]) \to C \), which looks pretty much like the result of induction hypothesis from above but not quite, because we cannot just randomly add a focus. We just don’t know if \( A \) is positive, and \( B \) negative. In fact, we would make a mistake as the following counter example shows. Let \( \Delta = a^+, a^+ \to b^+ \) and \( A = b^+ \). Clearly, \( \cdot; (a^+, a^+ \to b^+) \Rightarrow b^+ \) is provable, in the unfocused
system. But $\vdash (a^+, a^+ \rightarrow b^+) \rightarrow b^+$ is not: Since $b^+$ is not in $\Delta$, focusing on $b^+$ in

$$\vdash (a^+, a^+ \rightarrow b^+) \rightarrow [b^+]$$

will fail!

Instead, we will need to do some cutting.

$$\begin{array}{c}
\Gamma; A \rightarrow [A] \quad \Gamma; [B] \rightarrow B \\
\hline
\Gamma; (A, [A \rightarrow B]) \rightarrow B
\end{array} \quad \begin{array}{c}
\text{idR} \\
\text{idL}
\end{array} \quad \Gamma; (A, A \rightarrow B) \rightarrow B \quad \begin{array}{c}
\text{focusL}
\end{array}
\Gamma; \Delta_1 \rightarrow A \\
\hline
\Gamma; (\Delta_1, A \rightarrow B) \rightarrow B
\end{array} \quad \begin{array}{c}
\text{cut}
\end{array}
\Gamma; (\Delta_2, B) \rightarrow C \\
\hline
\Gamma; (\Delta_1, \Delta_2, A \rightarrow B) \rightarrow C
\end{array} \quad \begin{array}{c}
\text{cut}
\end{array}$$

Finally, we address the question of persistant resources. Since last lecture, the derivability judgment refers to $\Gamma$ the context of persistant resources. We will show below, that there is no need to define a focus for $\Gamma$, the only change is that when copying a resources from $\Gamma$ into $\Delta$, we will put a rule immediately into focus. As a consequence, we will introduce one new axiom rule, because if we are in right focus we must be able to look up for a positive atom in $\Gamma$ without loosing right focus. This is also why the copy rule focuses only on non-positive atoms.

$$\Gamma; \Delta_1 \vdash [A] \quad \Gamma; A \vdash [A] \quad \text{copy, } A \neq P^+ \quad \Gamma; P^+: \cdot 
\vdash [P^+] \quad \text{pax!}$$

Two things until we are done. First, we need to give focused versions of the $!$ left and right rules. This is straightforward since we know that $!$ is a positive connective.

$$\begin{array}{c}
\Gamma; \cdot \vdash [A] \\
\hline
\Gamma; \cdot \vdash [\!A] \quad \text{!R}
\end{array} \quad \begin{array}{c}
\Gamma; (\Gamma, A); \delta \rightarrow \gamma \\
\Gamma; (\delta, \!A) \rightarrow \gamma \\
\hline
\Gamma; (\Gamma, \!A); \delta \rightarrow \gamma \\
\Gamma; \cdot \vdash [\!A] \quad \Gamma; \cdot \vdash [A] \\
\hline
\Gamma; (\Gamma, \!A); \delta \rightarrow \gamma \\
\Gamma; \cdot \vdash [A] \\
\hline
\Gamma; \delta \rightarrow \gamma
\end{array} \quad \text{cut!}$$

References

