The Logic of Conditionals

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Conditional Sentences

Examples

- *If* it walks like a duck and talks like a duck, *then* it must be a duck.
- *If* kangaroos had no tails, (*then*) they would topple over.
- *If* I gave up teaching, (*then*) I will loose my job.

General Form

\[
\text{If } P \text{ then } Q
\]

where *P* is the *antecedent* and *Q* is the *consequent.*
Lewis (1968) on Counterfactuals

First Paragraph of Introduction

‘If kangaroos had no tails, they would topple over’ seems to me to mean something like this: in any possible state of affairs in which kangaroos have no tails, and which resembles our actual state of affairs as much as kangaroos having no tails permits it to, the kangaroos topple over. I shall give a general analysis of counterfactual conditionals along these lines.

This lecture

We look at Lewis’ account, but also at other possibilities to interpret conditionals.
Material Implication

- think $\rightarrow$ in propositional logic
- if $P$ then $Q \equiv P$ is false or $Q$ is true

Question

Is this the *right* reading of conditionals?

Yes – some good things are true

Modus Ponens $P \rightarrow Q, P \models Q$

Modus Tollens $P \rightarrow Q, \neg Q \models \neg P$

Disjunctive Syllogism $P \lor Q \models \neg P \rightarrow Q$
Problems with Material Implication

Truth of the Consequent

\[ P \models Q \rightarrow P \]

Dirk teaches Logic at 16:30. Therefore, if Dirk dies at 16:00, Dirk teaches Logic at 16:30.

Strengthening

\[ P \rightarrow Q \models P \land P' \rightarrow Q \]

If the Logic classes run smoothly, Dirk is happy. Therefore, if the Logic classes run smoothly and the coffee machine breaks, Dirk is happy.
More Problems with Material Implication

Transitivity

\[ P \rightarrow Q, Q \rightarrow R \models P \rightarrow R \]

If Dirk quits teaching, he won't be able to afford his house. If Dirk wins the lottery, he will quit teaching. Therefore, if Dirk wins the lottery, he will not be able to afford his house.

Contraposition

\[ P \rightarrow Q \models \neg Q \rightarrow \neg P \]

(Even) if Magellan would not have died in 1521, he would (still) not be alive today. Therefore, if Magellan were alive today, he would have died in 1521.
Conclusion and Analysis

Conditionals as Material Implication?

*If material implication does not provide the right semantics for conditionals, then we need to look elsewhere.*

(can this be modelled by material implication?)

Limits of Truth-Functionality

- If Dirk were in London, he would be in America.
- If Dirk were in London, he would be in Europe.

As Dirk is here, antecedent and consequent of both conditionals are false. But the second one is true whereas the first one isn’t.

Interpretation of Conditionals

*If we cannot account for conditionals in a truth-functional way, then we need to resort to a possible world semantics.*

(...and this?)
Semantics of Conditionals

There’s more than one type of conditional . . .

*The nice thing about standards is that you have so many to choose from.* (A. Tanenbaum)

(and the same goes for conditionals.)

Four main semantics

- as material conditionals (as we looked at)
- as strict conditionals (using necessity operators)
- as variably strict conditionals (using selection functions)
- as similarity conditionals (using spheres or onions)
Strict Conditionals

Recall Kripke Semantics and Modal Logic

If $P$ then $Q \equiv \Box(P \rightarrow Q)$ (write this as $P \Rightarrow Q$)

where $\Box$ is interpreted over standard Kripke models.

Informal Reading

It is necessarily the case that $P$ implies $Q$.

Truth of the Consequent

Do we have that $P \models (Q \Rightarrow P)$

that convinced us not to use material implication?
Shortcomings of Strict Conditionals

**Strengthening**

We have that

\[ P \Rightarrow Q \equiv P \land P' \Rightarrow Q \]

that we considered undesirable earlier (draw a Kripke model!).

**Contraposition**

We also have

\[ P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P \]

(give a proof!)

**Exercise**

Check all the other undesirable features of material implication that we looked at earlier on!
Main Idea: Selection Functions

We fix a set $W$ of worlds (as we cannot be truth-functional).

Take a function

$$f : W \times \mathcal{P}(W) \to W$$

that selects, at every world $w$, the world $f(w, [P])$ most similar to $w$ where $P$ is true.

Stipulate that

$$w \models P > Q \iff f(w, [P]) \models Q$$

where $[\cdot]$ denotes truth-set and $>$ the conditional.
Back to the Paradoxes

Strengthening

It is \textit{false} that

\[ P > Q \equiv P \land P' > Q \]

(draw a selection function model). The same goes for truth of the consequent.

Exercise

Check all the other paradoxes!

Critique: requires unique most similar worlds

- there may be more than one (think: real numbers)
- there may be none (think: false as antecedent)
Similarity Conditionals

Notation

We write $P \square \rightarrow Q$ for the similarity conditional, read as ‘if $P$ were true, then $Q$ would be true.’

Main Idea

Take a set $W$ of worlds, and for each world $w$, a *system of spheres* $S_w$ around $w$.

(Worl ds similar to $w$ congregate in smaller spheres).

Stipulate that

$$w \models P \square \rightarrow Q \iff ([P] \cap \bigcup S_w = \emptyset) \text{ or } (\emptyset \neq S \cap [P] \subseteq [Q])$$

for some $S \in S_w$, i.e. either $P$ is not entertainable, or some sphere makes $P \rightarrow Q$ true and contains at least one $P$ world.
(A) VACUOUS TRUTH

(B) NON-VACUOUS TRUTH

\[ \phi \square \rightarrow \psi \]
\[ \phi \square \rightarrow \sim \psi \]
\[ \phi \square \rightarrow \psi \]
\[ \sim (\phi \square \rightarrow \sim \psi) \]
(C) FALSITY - OPPOSITE TRUE

(D) FALSITY - OPPOSITE FALSE
(A) FAILURE OF TRANSITIVITY

\[ \psi \sim \psi \]

\[ \chi \quad \phi \]

\[ \phi \quad \psi \]

\[ \sim (\chi \quad \psi) \]
(B) FAILURE OF CONTRAPOSITION

\[ \phi \square \rightarrow \psi \]
\[ \sim (\sim \psi \square \rightarrow \sim \phi) \]
Axiomatisation

Lewis advocates the logic \( \text{VC} \) over *centered* sphere models:

**Centered Spheres Models**

A *sphere model* is a triple \((W, \$, V)\) where \(W\) is a set of worlds, \(\$\) is a *system of spheres*, i.e. \(\$_w\) is a nested system of subsets of \(W\), closed under unions and nonempty intersections for every world \(w \in W\), and \(V\) is a valuation of propositional constants.

A sphere model is *centered* if \(\{w\} \in \$_w\) for all \(w \in W\).

Truth condition is

\[ w \models P \Box \rightarrow Q \iff ([P] \cap \bigcup \$_w = \emptyset) \text{ or } (\emptyset \neq S \cap [P] \subseteq [Q]) \]

for some \(S \in \$_w\).
The Usual Questions

Questions

- Give a complete axiomatisation for VC
- What is the complexity of VC?
- (Does it have the interpolation property? . . .)

Lewis Axiomatisation

Thm: Lewis’ axiomatisation below is strongly complete.

Rules:

1. Modus Ponens,
2. Deduction within Conditionals: for any $n \geq 1$,
   \[
   \frac{\vdash \chi_1 \land \ldots \land \chi_n \models \psi}{\vdash ((\phi \rightarrow \chi_1) \land \ldots \land (\phi \rightarrow \chi_n)) \models (\phi \rightarrow \psi)}
   \]
3. Interchange of Logical Equivalents;

Axioms:

1. Truth-functional tautologies,
2. Definitions of non-primitive operators,\dagger
3. $\phi \rightarrow \phi$,
4. $(\sim \phi \rightarrow \phi) \rightarrow (\psi \rightarrow \phi)$,
5. $(\phi \rightarrow \sim \psi) \lor ((\phi \land \psi) \rightarrow \chi) \equiv (\phi \rightarrow (\psi \lor \chi))$,
6. $(\phi \rightarrow \psi) \rightarrow (\phi \supset \psi)$,
7. $(\phi \land \psi) \rightarrow (\phi \rightarrow \psi)$.

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Outlook

This Lecture

- just a small glimpse on conditional logics
- there are many variations, interpretations etc.
- even for sphere models, there are 26 different conditions and logics

Interesting Research Questions

- combine with vagueness (if you finish *quickly* ...)
- complexity of conditional reasoning

Literature and Examples: Mostly from Lewis’ Book on Counterfactuals.