Australian National University

Finite Model Theory

Qing Wang
qing.wang@anu.edu.au

Logic and Computation
Research School of Computer Science

Overview of Topics

- **Part 1**: – **Introduction**
  1. What is finite model theory?
  2. Connections to some areas in CS
     - Database theory
     - Complexity theory
  3. Basic definitions and terminology
  4. Inexpressibility proofs
  5. Classical results over finite structures
     - Failures
     - Successes
     - Open questions

Finite Model Theory

- Classical model theory concentrates on **all structures** - the origin is in mathematics.
  - Boolean algebras, random graphs, algebraically closed fields, various models of arithmetic, etc.
- Finite model theory studies logics over **finite structures** - the origin is in computer science.
  - Finite relations
  - Finite graphs
  - Finite strings
  - Finite classes of arithmetic structures
  - ...

Connections to Database Theory

- Finite model theory plays a central role in the development of database theory.
  - A database can be naturally viewed as a finite structure, e.g.,

Database	Structure
Relational databases	finite relations
XML databases	finite trees
Graph databases	finite graphs

  - A query language is often measured in terms of logic, e.g.,

Query language	Logic
Relational Calculus	FO
Datalog$\neg$	$\exists$LFP
Basic SQL + aggregation	FO + counting extension
Core XPath	MSO

- Database theory supplies finite model theory with key motivations and problems.

Connections to Database Theory

- **Example**: Reachability queries
  - **Q1.** Find pairs of cities (s, d) such that one can fly from s to d with at most one stop.
  - **Q2.** Find pairs of cities (s, d) such that one can fly from s to d with at most two stops.
  - **Q3.** Find pairs of cities (s, d) such that one can fly from s to d.

FLIGHTS	
Source	**Destination**
Berlin	Beijing
Beijing	Auckland
Auckland	Sydney

Connections to Database Theory

- **Example**: Reachability queries
  - **Q1.** Find pairs of cities (s, d) such that one can fly from s to d with at most one stop.

$$\{(x_s, x_d) \mid \textsc{Flights}(x_s, x_d) \\ \vee \exists x_1.(\textsc{Flights}(x_s, x_1) \wedge \textsc{Flights}(x_1, x_d))\}$$

FLIGHTS	
Source	**Destination**
Berlin	Beijing
Beijing	Auckland
Auckland	Sydney

ANSWER	
Berlin	Beijing
Beijing	Auckland
Auckland	Sydney
Berlin	Auckland
Beijing	Sydney

Connections to Database Theory

- **Example**: Reachability queries
  - **Q2.** Find pairs of cities (s, d) such that one can fly from s to d with at most two stops.

$$\{(x_s, x_d) \mid \textsc{Flights}(x_s, x_d) \\ \vee \exists x_1.(\textsc{Flights}(x_s, x_1) \wedge \textsc{Flights}(x_1, x_d)) \\ \vee \exists x_1, x_2.(\textsc{Flights}(x_s, x_1) \wedge \textsc{Flights}(x_1, x_2) \wedge \textsc{Flights}(x_2, x_d))\}$$

FLIGHTS	
Source	**Destination**
Berlin	Beijing
Beijing	Auckland
Auckland	Sydney

ANSWER	
Berlin	Beijing
Beijing	Auckland
Auckland	Sydney
Berlin	Auckland
Beijing	Sydney
Berlin	Sydney

Connections to Database Theory

- **Example**: Reachability queries
  - **Q3.** Find pairs of cities (s, d) such that one can fly from s to d.

Connections to Database Theory

- **Example**: **Reachability queries**
  - **Q3.** Find pairs of cities (s, d) such that one can fly from s to d.

$$\{(x_s, x_d) \mid \textsc{Flights}(x_s, x_d) \\ \lor \exists x_1.(\textsc{Flights}(x_s, x_1) \land \textsc{Flights}(x_1, x_d)) \\ \lor \exists x_1, x_2.(\textsc{Flights}(x_s, x_1) \land \textsc{Flights}(x_1, x_2) \land \textsc{Flights}(x_2, x_d)) \\ \lor \exists x_1, x_2, x_3.(\textsc{Flights}(x_s, x_1) \land \cdots \land \textsc{Flights}(x_3, x_d)) \\ \lor \ldots\}$$

Connections to Database Theory

- **Example**: **Reachability queries**
  - **Q3.** Find pairs of cities (s, d) such that one can fly from s to d.

$$\{(x_s, x_d) \mid \textsc{Flights}(x_s, x_d) \\ \lor \exists x_1.(\textsc{Flights}(x_s, x_1) \land \textsc{Flights}(x_1, x_d)) \\ \lor \exists x_1, x_2.(\textsc{Flights}(x_s, x_1) \land \textsc{Flights}(x_1, x_2) \land \textsc{Flights}(x_2, x_d)) \\ \lor \exists x_1, x_2, x_3.(\textsc{Flights}(x_s, x_1) \land \cdots \land \textsc{Flights}(x_3, x_d)) \\ \lor \ldots\}$$

Cannot be expressed in Relational Calculus (FO)

- *How can we tell whether a query CAN or CANNOT be expressed in a query language?*

Connections to Complexity Theory

- Computational complexity measures the amount of resources (e.g., time and space) that are needed to solve a problem.
  - Many models of computation have been invented, e.g.,

Models of Computation	
Lambda calculus	Church
Recursive functions	Gödel
Turing machines	Turing

  - Two key questions:
    - What can be automatically computed (i.e., computability)?
      ↪ Turing/Church thesis
    - How difficult it is to solve a problem (i.e., complexity)?
      ↪ Complexity classes (P, NP, Pspace,. . .)

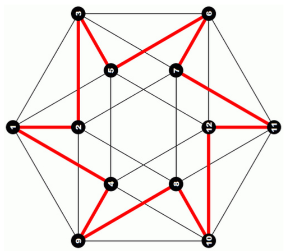
Connections to Complexity Theory

- The development of descriptive complexity is one of the most striking results in finite model theory.
- **How are different logics and complexity classes related?**
  1. What logic can be used to express a query?
  2. What is the complexity of evaluating a query?
- Some known results:

Query	Logic	Complexity class
Transitive closure	IFP + $<$	P
Connectivity	IFP + $<$	P
Evenness	IFP + $<$	P
Hamiltonicity	$\exists$SO	NP
3-colorability	$\exists$SO	NP
Clique	$\exists$SO	NP
Quantified SAT	PFP + $<$	Pspace

Connections to Complexity Theory

- **Example**: Hamiltonicity
  - A **Hamiltonian cycle** is a cycle that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**.



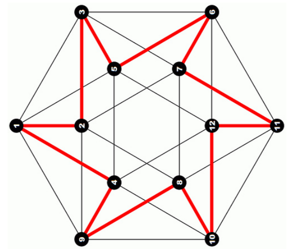
Connections to Complexity Theory

- **Example**: Hamiltonicity
  - A **Hamiltonian cycle** is a cycle that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a **Hamiltonian graph**.

$\exists L \exists S$ linear order$(L) \wedge$
S is the successor relation of $L \wedge$
$\forall x \exists y (L(x,y) \vee L(y,x)) \wedge$
$\forall x \forall y (S(x,y) \Rightarrow E(x,y))$

L is a linear ordering relation.

S is a circular successor relation:
$\forall x \forall y S(x,y) \Leftrightarrow$
$((L(x,y) \wedge \neg \exists z (L(x,z) \wedge L(z,y))) \wedge$
$(\neg \exists z L(x,z) \wedge \neg \exists z L(z,y)))$



Connections to Complexity Theory

- **Example**: Hamiltonicity
  - Hamiltonicity can be specified in *existential second-order logic* ($\exists$SO).

$\exists L \exists S$ linear order$(L) \wedge$
S is the successor relation of $L \wedge$
$\forall x \exists y (L(x,y) \vee L(y,x)) \wedge$
$\forall x \forall y (S(x,y) \Rightarrow E(x,y))$

  - Testing Hamiltonicity is an NP-complete problem.
  - So, *is there any connection between $\exists$SO and NP?*

Connections to Complexity Theory

- **Example**: Hamiltonicity
  - Hamiltonicity can be specified in *existential second-order logic* ($\exists$SO).

$\exists L \exists S$ linear order$(L) \wedge$
S is the successor relation of $L \wedge$
$\forall x \exists y (L(x,y) \vee L(y,x)) \wedge$
$\forall x \forall y (S(x,y) \Rightarrow E(x,y))$

  - Testing Hamiltonicity is an NP-complete problem.
  - So, *is there any connection between $\exists$SO and NP?*

$\exists$SO	Existential SO quantifiers	+	FO formula
NP	Guess stage	+	Verify stage

  - **Descriptive complexity** aims to characterize complexity classes by means of logics.

Finite Model Theory

- Apparently, finite structures are a subclass of “all structures”. Is finite model theory just a subfield of classical model theory?
- Some history:
  - Before 1970s, some problems about FO over finite structures were studied.
  - In 1970s,
    - Fagin published several papers relating to finite model theory.
    - Hájek proposed to “develop logic (classical and generalized) modified by allowing only finite models”.
    - ...
  - Since 1980s, finite model theory becomes an active line of research.

Finite Model Theory – Central Themes

- **Logic and expressiveness**
  - Ehrenfeucht-Fraïssé games
  - Locality
- **Logic and complexity**
  - Descriptive complexity
  - Data complexity and expression complexity
- **Logic and combinatorics**
  - Zero-one laws
  - Asymptotic probabilities
- **Various logics**
  - Finite variable logics
  - Fixed point logics
  - Logics with counting
  - ...

Structures

- A **vocabulary** σ is a *finite* set of relation symbols, each with a fixed arity.

  Note that, we restrict vocabularies to be relational, and there are *no function symbols* in σ.
- A **structure** (also called a **model**) of vocabulary σ is $\mathfrak{A} = \langle A, (R^{\mathfrak{A}})_{R \in \sigma} \rangle$, where
  - A, called the **universe** of $\mathfrak{A}$, is a nonempty set, and
  - each $R^{\mathfrak{A}} \subseteq A^n$ is an interpretation of a n-ary relation symbol from σ.
- A structure $\mathfrak{A}$ is called **finite** if its universe A is a finite set.
- *Conventions:*
  - We denote universes by using Roman letters corresponding to their structures, e.g., the universe of $\mathfrak{A}$ is A, the universe of $\mathfrak{B}$ is B, etc.
  - We use the same symbol R for both a relation symbol in σ, and its interpretation $R^{\mathfrak{A}}$.

First-order Logic

- Recall terms and formulas of first-order logic (FO) (we assume a countably infinite set of variables).
- **Syntax**:
  - **Terms** of FO are defined by:
    - Each variable x and constant c is a term.
    - $f(t_1, \ldots, t_n)$ is a term, where f is a relation symbol, and $t_1, \ldots, t_n$ are terms.
  - **Formulas** of FO can be inductively defined by:
    - atomic formulas: $R(t_1, \ldots, t_n)$, $t_1 = t_2$;
    - Boolean operations: $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\neg\varphi$;
    - first-order quantifiers: $\exists\varphi$, $\forall\varphi$.
- **Semantics**: we skip the details as it has been covered in the last week

First-order Logic

- A **sentence** is a formula without free variables.
- A sentence φ is **satisfiable** if it has a model, and is **valid** if it is true in every structure.
  - φ is not valid iff $\neg\varphi$ is satisfiable.
- A sentence φ is **finitely satisfiable** if it has a finite model, and is **finitely valid** if it is true in every finite structure.
  - φ is not finitely valid iff $\neg\varphi$ is finitely satisfiable.
- We use $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$ to denote that $\varphi(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$.

Queries

- A formula $\varphi(x_1, \ldots, x_n)$ with free variables $x_1, \ldots, x_n$ defines **a mapping** Q that associates to every structure $\mathfrak{A}$ a (n-ary) relation on $\mathfrak{A}$:

$$Q(\mathfrak{A}) = \{(a_1, \ldots, a_n) | \mathfrak{A} \models \varphi(a_1, \ldots, a_n)\}.$$

- A **n-ary query** Q is such a mapping **closed under isomorphism**, i.e., if $h : A \to B$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, it is also an isomorphism between $Q(\mathfrak{A})$ and $Q(\mathfrak{B})$.
- If $n = 0$, a 0-ary query is a mapping from structures to $\{true, false\}$, which is called a **Boolean query**.
- A query Q is **definable** in a logic L if there is a formula $\varphi(x_1, \ldots, x_n)$ of L that defines Q. A Boolean query Q is **definable** in a logic L if there is a formula φ of L such that $Q(\mathfrak{A}) = true$ iff $\mathfrak{A} \models \varphi$ for all $\mathfrak{A}$.

Queries Definable in FO

- Consider $\sigma = \{E\}$ and $G = \langle V, E \rangle$.

1. Graphs whose edges are antireflexive and symmetric:
$$\forall x \neg E(x, x) \land \forall x \forall y (E(x, y) \Rightarrow E(y, x)).$$
2. Graphs that contain at least one triangle:
$$\exists x \exists y \exists z (x \neq y \land y \neq z \land x \neq z \land E(x, y) \land E(x, z) \land E(y, z)).$$
3. Graphs that contain at least n vertices:
$$\exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j.$$

Limitations of FO

- The expressive power of FO on finite structures is limited:
  - Cannot express counting properties, e.g.,
    1. **Evenness**: Given a graph G, is the number of vertices in G even?
$$even(V) = \begin{cases} 1 & \text{if } |V| \text{ is even} \\ 0 & \text{otherwise .} \end{cases}$$
  - Cannot express properties that require iterative algorithms, e.g.,
    1. **Connectivity**: Given a graph G, is it connected?
       I.e., there exists a path between any two nodes a and b in G.
  - Cannot express properties that require to quantify relations, e.g.,
    1. **3-Colorability**
    2. **Clique**
    3. ...

Inexpressibility Proofs

- *How can one prove that a property is inexpressible in a logic, e.g., FO?*
- Some techniques are available for inexpressibility proofs in FO.
  - Compactness theorem
  - Ehrenfeucht-Fraïssé games
  - ...

Inexpressibility Proofs

- **Compactness theorem**

  Let Φ be a set of first-order sentences. If every finite subset of Φ is satisfiable, then Φ is satisfiable.
- Main ideas of using the compactness theorem to prove inexpressibility of a property P:
  - Assume P is expressible by a FO-sentence φ.
  - Construct a set of sentences Ψ so that each model of Ψ does not satisfy P, but each finite subset of Ψ has a model satisfying P.
  - By compactness we would know that $\Psi \cup \{\varphi\}$ has a model.
  - Contradiction with the assumption.

Inexpressibility Proofs

- **Connectivity** of *arbitrary graphs* is not FO-definable.

  Proof:
  - Assume that connectivity is definable by ϕ.
  - Expand the vocabulary with two constants c_1 and c_2, and let $T = \{\psi_n | n > 0\} \cup \{\phi\}$, where
    $$\psi_n = \neg(\exists x_1 ... \exists x_n x_1 = c_1 \wedge x_n = c_2 \wedge \bigwedge_{1 \le i \le n-1} E(x_i, x_{i+1}))$$
    i.e., there is no path of length n+1 from c_1 to c_2.
  - Every finite subset $T' \subseteq T$ is satisfiable, because there exists N, s.t. for all $\psi_n \in T'$, $n < N$, and T' has a model with a path of length N+1. By compactness, T is satisfiable.
  - However, T has no model. *Contradiction.*

Inexpressibility Proofs

- *Does the previous proof tell us that FO cannot express connectivity over finite graphs*?

  The previous proof:
  - Assume that connectivity is definable by ϕ.
  - Expand the vocabulary with two constants c_1 and c_2, and let $T = \{\psi_n | n > 0\} \cup \{\phi\}$, where
    $$\psi_n = \neg(\exists x_1 ... \exists x_n x_1 = c_1 \wedge x_n = c_2 \wedge \bigwedge_{1 \le i \le n-1} E(x_i, x_{i+1}))$$
    i.e., there is no path of length n+1 from c_1 to c_2.
  - Every finite subset $T' \subseteq T$ is satisfiable, because there exists N, s.t. for all $\psi_n \in T'$, $n < N$, and T' has a model with a path of length N+1. By compactness, T is satisfiable.
  - However, T has no model. *Contradiction.*

Failure of Compactness Theorem

- *Does the previous proof tell us that FO cannot express connectivity over finite graphs*?
- To modify the previous proof for finite models, one would be to use compactness over finite models.
- But compactness fails over finite models.

  Proposition: There is a set T of FO-sentences s.t.

  1. T has no finite models, and
  2. every finite subset of T has a finite model.

Failure of Compactness Theorem

- **Proposition**: There is a set T of FO-sentences s.t.
  1. T has no finite models, and
  2. every finite subset of T has a finite model.

Proof:

- Assume that $\sigma = \emptyset$, and define

$$\psi_n = \exists x_1 ... \exists x_n \bigwedge_{i \neq j} \neg(x_i = x_j).$$

  i.e., the universe has at least n distinct elements.
- Let $T = \{\psi_n | n > 0\}$.
- T has no finite model. But for each finite subset of T, a set whose cardinality exceeds the maximal n is a model.

Inexpressibility Proofs

- For the techniques available for inexpressibility proofs in FO:
  - **Compactness theorem**
    ↪ fails over finite structures.
  - **Ehrenfeucht-Fraïssé games**
    ↪ used as a central tool on classes of finite structures.

Classical Model Theory

- Main topics:
  - Logical definability of structural properties
  - Classification of models of theories
  - Algebraic properties in axiomatic theories
- Some key results:
  1. Completeness theorem (and Compactness theorem)
  2. Löwenheim-Skolem theorem
  3. Beth's definability theorem
  4. Craig's interpolation theorem
  5. Los-Tarski preservation theorem and Lyndon's positivity theorem

Classical Model Theory

- Some key results:
  1. Completeness theorem (and Compactness theorem)
  2. Löwenheim-Skolem theorem
  3. Beth's definability theorem
  4. Craig's interpolation theorem
  5. Los-Tarski preservation theorem and Lyndon's positivity theorem
- *Can these results of classical model theory hold over finite structures?*
- Unfortunately, **all the above results fail when we restrict to finite structures.**

Failure of Completeness Theorem

- **Trakhtenbrot's Theorem**[1]

  For every relational vocabulary σ with at least one binary relation symbol, it is undecidable whether a sentence Φ of σ is finitely satisfiable.

  Basic idea:
  - For any Turing machine M, there is a FO sentence φ_M such that

    M halts iff φ_M has a finite model.
- **Corollary**

  For every relational vocabulary σ with at least one binary relation symbol, **the set of finitely valid FO sentences is not recursively enumerable**.

  Basic idea:
  - Because the set of non-halting Turing machines is not recursively enumerable, $\{\varphi | \neg\varphi$ has no finite models$\}$ is also not recursively enumerable.

[1] B. Trakhtenbrot, The Impossibility of an Algorithm for the Decidability Problem on Finite Classes. Proceedings of the USSR Academy of Sciences (in Russian) 70 (4): 569572, 1950.

Failure of Completeness Theorem

- Some consequences of Trakhtenbrot's Theorem:
  - Unsatisfiability of FO is semi-decidable, but finite unsatisfiability is not semi-decidable.
  - Satisfiability of FO is not semi-decidable, but finite satisfiability is semi-decidable.

Failure of Löwenheim-Skolem Theorem

- **Corollary**

  There is no recursive function f s.t. if a FO sentence φ has a finite model, then it has a model of size at most $f(\varphi)$.
- Recall the Löwenheim-Skolem theorem in classical model theory:

  If a countable first-order theory has an infinite model, then for every infinite cardinal number k it has a model of size k.

Trakhtenbrot Theorem

- **Theorem** (Trakhtenbrot, 1950)

  For every relational vocabulary σ with at least one binary relation symbol, it is undecidable whether a sentence Φ of σ is finitely satisfiable.

- **Proof**:

  For every Turing machine $M = (S, \Sigma, \Delta, \delta, q_0, S_a, S_r)$, construct a sentence φ_M of σ s.t. φ_M **is finitely satisfiable iff** M **halts on the empty input**.

  - Let $\sigma = \{<, min, T_0, T_1, (H_q)_{q \in S}\}$, and φ_M states that each relation symbol in σ is interpreted below, and M eventually halts.
    - $<$ is a linear order, and *min* is the minimal element w.r.t. $<$;
    - $T_0(p, t)$ and $T_1(p, t)$ are *tape predicates*: indicate that position p at time t contains 0 or 1;
    - $H_q(p, t)$'s are *head predicates*: indicate that at time t, M is in state q and its head is in position p.
  - If φ_M has a finite model, then such a model represents a computation of M that halts on an empty input, and vice versa.

Classical Result – Revisited for Finite Models

- *Are there any results of classical model theory that survive on finite models?*
- Rosen wrote:[2]

  > *"there seems to be no example of a theorem [of classical model theory] that remains true when relativized to finite structures but for which there are entirely different proofs for the two cases. It would be interesting to find a theorem proved using the compactness theorem that can be established using a new method over finite structures."*

[2] Rosen, E. Some aspects of model theory and finite structures. Bull. Symbolic Logic 8, 380-403, 2002.

Classical Result – Revisited for Finite Models

- A FO sentence $\varphi(\vec{x})$ is **preserved under homomorphisms** if $\mathfrak{A} \models \varphi(\vec{a})$ implies $\mathfrak{B} \models \varphi(h(\vec{a}))$ whenever $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism.

- Recall the homomorphism preservation theorem in classical model theory:

  The following statements are equivalent for any FO sentence φ:

  1. φ is preserved under homomorphisms on all structures;
  2. $\varphi \equiv \varphi^*$ on all structures for some existential positive FO-sentence φ^*.

- **Theorem** (Rossman, 2005)

  If a FO sentence φ is preserved under homomorphisms on all finite structures, then there is an existential positive FO-sentence φ^* that is equivalent to φ on all finite structures.

Classical Result – Revisited for Finite Models

- First-order logic has a special place in classical model theory.

- **Theorem** (Lindström, 1969)

  First-order logic is a maximal logic possessing both the Compactness Theorem and the Löwenheim-Skolem Theorem, i.e., no logic that is compact and satisfies the Löwenheim-Skolem property can properly extend FO.

- **Question**
  - Is there a similar characterization of first-order logic/least fixed-point logic on finite structures?

Useful References

1. L. Libkin. Elements of Finite Model Theory. Springer, 2004.
2. M. Grohe. Descriptive Complexity, Canonisation, and Definable Graph Structure Theory, 2011.
3. H.-D. Ebbinghaus and J. Flum. Finite Model Theory. Springer, 2005.
4. N. Immerman. Descriptive Complexity. Springer, 1999.
5. M. Otto. Finite Model Theory, available at http://www.mathematik.tu-darmstadt.de/ otto/LEHRE/FMT0809.ps
6. Abiteboul, Hull, Vianu. Foundations of Databases. 1995
7. Grädel et al. Finite Model Theory and Its Applications. 2007