Overview of Topics

Part 2: – Ehrenfeucht-Fraïssé Games

- Elementary equivalence and isomorphism
- Ehrenfeucht-Fraïssé (EF) games
  - Rules
  - Winning strategies
- Partial isomorphism
- Equivalence relation
- EF theorem
- EF applications

Recall - Inexpressibility Proofs

- How can one prove that a property $P$ is inexpressible in a logic $L$ on a class $C$ of structures?
  - To prove that $P$ is expressible, one needs to find a formula of $L$ that defines $P$ on $C$.
  - To prove that $P$ is not expressible, one has to show no formula of $L$ that defines $P$ on $C$.

- Common techniques used for inexpressibility proofs in first-order logic:
  - Compactness theorem
    - fails over finite structures.
  - Ehrenfeucht-Fraïssé games
    - used as a central tool on classes of finite structures.

Elementary Equivalence and Isomorphism

- Elementary equivalence, formulated by Alfred Tarski, is an important model-theoretic notion.
- Two models $\mathfrak{A}$ and $\mathfrak{B}$ over the same vocabulary are **elementarily equivalent** if, for every first-order sentence $\varphi$, $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.
  That is, if two models are elementarily equivalent, then they cannot be distinguished by any first-order sentence.
- The notion of elementary equivalence is important to establishing inexpressibility results.
  - First, prove that two models are elementarily equivalent.
  - Then, show that a property $P$ that can distinguish the two models.
  - Thus, the property $P$ is not definable.
Elementary Equivalence and Isomorphism

- Two models $\mathfrak{A}$ and $\mathfrak{B}$ over the same vocabulary are **isomorphic** if there is a bijective mapping $h: A \rightarrow B$ preserving relations and constants.

- In general, two isomorphic models must be elementarily equivalent, but two elementarily equivalent models are not necessarily isomorphic.

Elementary Equivalence and Isomorphism

**Theorem**

For every finite structure $\mathfrak{A}$, there is a first-order sentence $\varphi$ such that $\mathfrak{B} \models \varphi$ iff an arbitrary structure $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$.

**Proof**

- Assume w.l.o.g. that $\mathfrak{A}$ is a graph $(V, E)$ where $V = \{a_1, \ldots, a_n\}$.
- Define $\varphi$ as
  
  $\exists x_1 \ldots \exists x_n ((\forall y \setminus \{x_i\} \lnot (x_i = x))$  
  $\land (\forall y \lor y = x_i))$  
  $\land (\bigwedge_{\{a_i, a_j\} \in E} E(x_i, x_j))$  
  $\land (\bigwedge_{\{a_i, a_j\} \notin E} \lnot E(x_i, x_j))$

- We have $\mathfrak{A} \models \varphi$. If $\mathfrak{B} \models \varphi$, then $\mathfrak{B}$ is isomorphic to $\mathfrak{A}$.

Methodology for Inexpressibility Proofs

- Thus, for finite structures, the notion of elementary equivalence is **too strong** to establishing inexpressibility results.
- One way to solve this is to weaken the relation of elementary equivalence by stratifying formulas in a logic.
To prove that a property \( P \) is not expressible in a logic \( L \) over finite structures, we can do the following:

- Partition the set of all formulas of \( L \) into countably many classes, i.e., \( L[0], L[1], \ldots, L[k], \ldots \);
- Find two families of structures \( \{ A_k | k \in \mathbb{N} \} \) and \( \{ B_k | k \in \mathbb{N} \} \) such that
  - \( A_k \models \varphi \) iff \( B_k \models \varphi \) for every sentence \( \varphi \) in \( L[k] \); and
  - \( A_k \) has property \( P \), but \( B_k \) does not.

But...
- How to partition FO into such classes?
- How to show that two families of structures agree on classes of FO?

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Quantifier Rank

- The quantifier rank of a formula \( \varphi \), written as \( qr(\varphi) \), is its depth of quantifier nesting, i.e.,
  - If \( \varphi \) is atomic, then \( qr(\varphi) = 0 \).
  - \( qr(\varphi_1 \land \varphi_2) = \max(qr(\varphi_1), qr(\varphi_2)). \)
  - \( qr(\neg \varphi) = qr(\varphi) \).
  - \( qr(\exists x \varphi) = qr(\forall x \varphi) = qr(\varphi) + 1 \).

**Example:** What is the quantifier rank of \( d_k \)? What is the total number of quantifiers in \( d_k \)?

- \( d_0(x, y) = E(x, y) \)
- \( \ldots \)
- \( d_k = \exists z d_{k-1}(x, z) \land d_{k-1}(z, y) \)

- The set of all FO-formulas is partitioned into many classes, denoted as \( FO[k] \), each having all formulas of quantifier rank up to \( k \).
Equivalence Relation

- We write $\mathfrak{A} \equiv_k \mathfrak{B}$ for two structures $\mathfrak{A}$ and $\mathfrak{B}$ iff the following equivalence holds for all sentences $\varphi \in FO[k]$:

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi,$$

i.e., $\mathfrak{A}$ and $\mathfrak{B}$ cannot be distinguished by FO sentences with $qr(\varphi) < k$.

- Let $\bar{a}$ and $\bar{b}$ be two tuples from $\mathfrak{A}$ and $\mathfrak{B}$, respectively. We write $(\mathfrak{A}, \bar{a}) \equiv_k (\mathfrak{B}, \bar{b})$ iff the following equivalence holds for all formulas $\varphi \in FO[k]$, where

$$\mathfrak{A} \models \varphi[\bar{a}] \iff \mathfrak{B} \models \varphi[\bar{b}]$$

- Note that,
  - $\mathfrak{A} \equiv_k \mathfrak{B}$ is a weakening of elementary equivalence by only considering the class of FO sentences/formulas of quantifier rank up to $k$.
  - $\equiv_k$ has finitely many equivalence classes, each of which is FO-definable.

Partial Isomorphism

- Recall that all finite structures are relational (no function symbols).

- Let $\mathfrak{A}_{|A'}$ be the substructure of $\mathfrak{A}$ to the subdomain $A' \subseteq A$, i.e., for each relation $R$:

$$R^{\mathfrak{A}_{|A'}} := \{(a_1, \ldots, a_n) \in R^A | a_1, \ldots, a_n \in A'\}.$$

- A partial function $\zeta : |A| \to |B|$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ if $\zeta$ is an isomorphism between $R^{\mathfrak{A}_{|\text{dom}(\zeta)}}$ to $R^{\mathfrak{B}_{|\text{rng}(\zeta)}}$.

Partial Isomorphism

- Are they partial isomorphisms?

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EF Games

- Ehrenfeucht-Fraïssé (EF) games:
  - Fraïssé was the first to find a purely structural necessary and sufficient condition for two structures to be elementarily equivalent (1954).
  - Ehrenfeucht reformulated this condition in terms of games (1961).

- One of the few model-theoretic techniques that apply to finite structures as well as infinite ones
  - The infinite case: a number of more powerful tools available
  - The finite case: a central tool for describing expressiveness of logics, e.g., measure the expressive power of database query languages

- Variations for capturing different logics-describing different equivalences

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**EF Games - Rules**

- Two structures $\mathcal{A}$ and $\mathcal{B}$ over the same vocabulary.
- Two players: **Spoiler, Duplicator**.
  - **Spoiler** tries to show that $\mathcal{A}$ and $\mathcal{B}$ are different.
  - **Duplicator** tries to show that $\mathcal{A}$ and $\mathcal{B}$ are the same.
- The players play a fixed number of rounds, each having three steps:
  - **Spoiler** picks a structure ($\mathcal{A}$ or $\mathcal{B}$).
  - **Spoiler** makes a move by picking an element of that structure.
  - **Duplicator** responds by picking an element in the other structure.
- After $n$ rounds, two sequences have been chosen:
  - $(a_1, \ldots, a_n)$ from $\mathcal{A}$.
  - $(b_1, \ldots, b_n)$ from $\mathcal{B}$.

**EF Games - Examples**

- Consider the following two structures:
  
  $\mathcal{A} = \langle \{a_1, a_2\}, \emptyset \rangle$
  
  $\mathcal{B} = \langle \{b_1\}, \emptyset \rangle$

- Some plays:

<table>
<thead>
<tr>
<th>Player</th>
<th>Choice</th>
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<tbody>
<tr>
<td>Spoiler</td>
<td>$a_1$</td>
</tr>
<tr>
<td>Duplicator</td>
<td>$b_1$</td>
</tr>
<tr>
<td>Spoiler</td>
<td>$a_2$</td>
</tr>
<tr>
<td>Duplicator</td>
<td>$b_1$</td>
</tr>
<tr>
<td>Spoiler</td>
<td>$a_3$</td>
</tr>
<tr>
<td>Duplicator</td>
<td>$b_1$</td>
</tr>
<tr>
<td>Spoiler</td>
<td>$a_4$</td>
</tr>
<tr>
<td>Duplicator</td>
<td>$b_1$</td>
</tr>
</tbody>
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**EF Games - Winning Strategies**

- How can **Spoiler** or **Duplicator** win in a game?
**EF Games - Winning Strategies**

- **Winning position:** **Duplicator** wins a run of the game if the mapping between elements of the two structures defined by the game run is a partial isomorphism. Otherwise, **Spoiler** wins.

- A player has an **n-round winning strategy** if s/he can play in a way that guarantees a winning position after n rounds, no matter how the other player plays.

- There is always either a winning strategy for **Spoiler** or for **Duplicator**.

- **Notation:**
  - $\mathcal{A} \sim_n \mathcal{B}$: if there is an n-round winning strategy for **Duplicator**.
  - $\mathcal{A} \not\sim_n \mathcal{B}$: if there is an n-round winning strategy for **Spoiler**.

Easy to see that $\mathcal{A} \sim_n \mathcal{B}$ implies $\mathcal{A} \sim_k \mathcal{B}$ for every $k \leq n$.

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**EF Games - Examples**

- Consider only 2 rounds of the EF game on $\mathcal{A} = \langle \{a_1, a_2\}, \emptyset \rangle$ and $\mathcal{B} = \langle \{b_1\}, \emptyset \rangle$.

    ![Game Diagram 1](image1)

- Consider only 2 rounds of the EF game on $\mathcal{A} = \langle \{a_1, a_2\}, \emptyset \rangle$ and $\mathcal{B} = \langle \{b_1\}, \emptyset \rangle$.

    ![Game Diagram 2](image2)

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**EF Games - Examples**

- Consider the EF game on $\mathcal{A} = \langle \{a_1, a_2\}, \emptyset \rangle$ and $\mathcal{B} = \langle \{b_1\}, \emptyset \rangle$.

    ![Game Diagram 3](image3)

- Is it a partial isomorphism?

    | Player     | Choice |
    |------------|--------|
    | Spoiler    | $a_1$  |
    | Duplicator | $b_1$  |
    | Spoiler    | $a_2$  |
    | Duplicator | $b_1$  |

- Who wins the plays?

- Duplicator has a winning position if $(S \leftrightarrow a_1, D \leftrightarrow b_1, S \leftrightarrow a_1, D \leftrightarrow b_1)$.
EF Games - Examples

- Consider only 2 rounds of the EF game on $\mathcal{A} = \{a_1, a_2\}$ and $\mathcal{B} = \{b_1\}$. 

Round 1

Round 2

- Spoiler has a winning position if $(S \leftrightarrow b_1, D \leftrightarrow a_1, S \leftrightarrow b_1, D \leftrightarrow a_2)$.

EF Games on Sets

- Let $\sigma = \emptyset$, and $\mathcal{A}$ and $\mathcal{B}$ be two sets of size at least $n$, i.e., $|\mathcal{A}|, |\mathcal{B}| \geq n$.

- Is it true that $\mathcal{A} \sim_n \mathcal{B}$?

EF Games - Examples

- Consider only 2 rounds of the EF game on $\mathcal{A} = \{a_1, a_2\}$ and $\mathcal{B} = \{b_1\}$. 

Round 1

Round 2

- Who has a 2-round winning strategy? Spoiler!
EF Games on Sets

- Let $\sigma = \emptyset$, and $\mathcal{A}$ and $\mathcal{B}$ be two sets of size at least $n$, i.e., $|\mathcal{A}|, |\mathcal{B}| \geq n$.
- Is it true that $\mathcal{A} \sim_n \mathcal{B}$?
- **Winning strategy** for Duplicator:
  - Suppose that the position is $((a_1, \ldots, a_i), (b_1, \ldots, b_i))$.
  - **Spoiler** picks an element $a_{i+1} \in A$:
    
    $$
    \begin{align*}
    \text{Duplicator} & \text{ picks } b_{i+1} = b_i & \text{if } a_{i+1} = a_i \text{ for } j \leq i \\
    \text{Duplicator} & \text{ picks } b_i \in B - \{b_1, \ldots, b_i\} & \text{otherwise}
    \end{align*}
    $$

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EF Games - Examples

- Consider 3 rounds of the EF game on $\mathcal{A} = \langle\{a_1, \ldots, a_4\}, \{E\}\rangle$ and $\mathcal{B} = \langle\{b_1, \ldots, b_5\}, \{E\}\rangle$.

  - Is it a partial isomorphism?

  - **Who wins the play?**

  - **Who has a 3-round winning strategy?**

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EF Games - Examples

- Consider 3 rounds of the EF game on $\mathcal{A} = \langle\{a_1, \ldots, a_4\}, \{E\}\rangle$ and $\mathcal{B} = \langle\{b_1, \ldots, b_5\}, \{E\}\rangle$.

  - **Who wins the play?**

  - **Who has a 3-round winning strategy?**
Consider 3 rounds of the EF game on $\mathcal{A} = \langle \{a_1, \ldots, a_3\}, \{E\}\rangle$ and $\mathcal{B} = \langle \{b_1, \ldots, b_3\}, \{E\}\rangle$.

Who has a 3-round winning strategy? Spoiler!

If we change $\sigma = \{E\}$ to $\sigma = \{<\}$ where $<$ is interpreted as a linear order, and consider the following two structures:

- $\mathcal{A} = \langle \{a_1, \ldots, a_4\}, \{<\}\rangle$
- $\mathcal{B} = \langle \{b_1, \ldots, b_5\}, \{<\}\rangle$

Consider the EF game on $\mathcal{A} = \langle \{a_1, \ldots, a_4\}, \{E\}\rangle$ and $\mathcal{B} = \langle \{b_1, \ldots, b_5\}, \{E\}\rangle$ again.

We know that Spoiler has a 3-round winning strategy now, but
- Who has a 1-round winning strategy?
- Who has a 2-round winning strategy?