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Finite Model Theory

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Overview of Topics

- **Part 3**: – **Logic and Complexity**
  1. Basics in complexity theory
  2. Descriptive complexity
  3. Some logics
     - Second-order logic ($\forall$SO, $\exists$SO)
     - Fixed Point Logics (LFP, IFP, PFP)
  4. Logical characterizations
     - NP and coNP
     - Ptime + $>$
     - Pspace + $>$
  5. Open problem: a logic for Ptime?

Complexity Theory

- Classify computational problems into different classes according to their inherent difficulty, and studying relationship among those classes.
- **Complexity class**: a set of languages determined by an abstract machine M using $O(f(n))$ of resource R, where n is the size of the input.
  - **An abstract machine** – a deterministic Turing machine, etc.
  - **Resource** – time, space, communications, processors, etc.
  - **Resource bound** – a function to bound the amount of resource.

$$\text{Ptime} \subseteq \begin{cases} \text{NP} \\ \text{coNP} \end{cases} \subseteq \text{Pspace}$$

Time Complexity

- Let M be a Turing machine (TM) that halts on all inputs.
- The **time complexity** of M is the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is **the maximum number of steps** that M uses on an input of length n.
  - A language L **is in** $DTIME(f(n))$ if L is accepted by a deterministic M and the running time of M is $O(f(n))$.
  - A language L **is in** $NTIME(f(n))$ if L is accepted by a nondeterministic M and the running time of M is $O(f(n))$.

Complexity Class – P

- **Ptime (P)** is the class of languages decidable by a **deterministic TM** in **polynomial time**:

$$\mathbf{P} = \bigcup_{k \in \mathbb{N}} DTIME(n^k).$$

- P plays a central role in complexity theory:
  1. mathematically robust class – invariant for models of computation;
  2. practically important class – realistically solvable on a computer.

Complexity Class – NP

- **NP** is the class of languages accepted by a **nondeterministic TM** in **polynomial time**:

$$\mathbf{NP} = \bigcup_{k \in \mathbb{N}} NTIME(n^k).$$

- **coNP** is the class of languages whose **complements** are in NP (i.e.,"no"-instances can be accepted in polynomial time by a non-deterministic TM).

$$\mathbf{P} \subseteq \mathbf{NP} \cap \mathbf{coNP}$$

It is not known:

- whether the containment is proper, and
- whether NP equals coNP.

Space Complexity

- Let M be a deterministic TM that halts on all inputs.
  - The **space complexity** of M is the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is **the maximum number of tape cells** that M uses on inputs of length n.
  - A language L **is in** $DSPACE(f(n))$ if L is decided by M and the running space of M is $O(f(n))$.

Complexity Class – Pspace

- **Pspace** is the class of languages decidable in polynomial-space on a deterministic TM:

$$\mathbf{Pspace} = \bigcup_{k \in \mathbb{N}} DSPACE(n^k).$$

- By Savitch's theorem, the nondeterministic case collapses to the deterministic one, in the case of space complexity.

P versus NP

$$P =? = NP$$

- P = the class of languages for which membership can be **decided** quickly.
- NP = the class of languages for which membership can be **verified** quickly.

Logic and Complexity

- *Can computational complexity theory be helpful to the logician? Can logic be used as a tool to obtain pure complexity-theoretic results?*
- **Logical definability**: logical resources needed to express queries $\rightsquigarrow$ lead to different logics.
  1. number of variables
  2. type of quantifiers
  3. restriction on negation
  4. ...
- **Computational complexity**: computational resources needed to compute queries $\rightsquigarrow$ lead to different complexity classes.
  1. time
  2. space
  3. ...

Descriptive Complexity

- Studies the connections between **computational complexity** and **logical definability**.
- By saying that **a logic L captures a complexity class C**, it means that
  1. Every query that can be evaluated in C over finite structures is definable in L
  2. Every query definable in L can be evaluated in C.
- Provides **machine-independent characterization** of complexity classes, without using any notion of machine, computation, or time.
- Allows us to convert a problem in complexity theory into an equivalent problem in logic, and vice versa.

Descriptive Complexity – Main Results

- Many computational complexity classes can be characterized in terms of logical definability on classes of finite structures.
  1. Fagin, 1974
     NP $\equiv \exists$SO on all finite structures
  2. Immerman and Vardi, 1982
     Ptime $\equiv$ LFP on all ordered finite structures
  3. Abiteboul and Vianu, 1982
     Pspace$\equiv$ PFP on all ordered finite structures
  4. ...
- Generally, there are two cases so far:
  - A complexity class (NP or higher) is characterized by a logic over all finite structures.
  - A complexity class (P or lower) is characterized by a logic over all ordered finite structures.

Second-Order Logic

- **Second-order logic** (SO) extends FO logic by allowing second-order quantifiers:

$$\exists X\varphi \text{ and } \forall X\varphi,$$

where X is a second-order variable, ranging over relations on the universe.

Let $\mathfrak{A}$ be a structure.

- $\mathfrak{A}$ satisfies $\exists X\varphi$ if, for *some* n-ary relation R on the universe of $\mathfrak{A}$, $\mathfrak{A} \models \varphi(X/R)$.
- $\mathfrak{A}$ satisfies $\forall X\varphi$ if, for *all* n-ary relations R on the universe of $\mathfrak{A}$, $\mathfrak{A} \models \varphi(X/R)$.

Let $\varphi(X_1, \ldots, X_k)$ be a FO formula.

- **Existential second-order logic** ($\exists$SO) has the form:
$$\exists X_1 \ldots \exists X_n \varphi(X_1, \ldots, X_n);$$
- **Universal second-order logic** ($\forall$SO) has the form:
$$\forall X_1 \ldots \forall X_n \varphi(X_1, \ldots, X_n),$$

Second-Order Logic - Examples

- **Colourability**: Color the vertices of a graph such that no two vertices sharing the same edge have the same color.

- A graph (V, E) is **2-colourable** iff the following formula is true.

$$\exists R \forall x \forall y \quad (E(x,y) \Rightarrow (R(x) \Leftrightarrow \neg R(y)))$$

- A graph (V, E) is **3-colourable** iff the following formula is true.

$$\begin{aligned}\exists R \exists B \exists G \quad & \forall x(R(x) \vee B(x) \vee G(x)) \wedge \\ & \forall x(\neg(R(x) \wedge B(x)) \wedge \neg(B(x) \wedge G(x)) \wedge \neg(R(x) \wedge G(x))) \wedge \\ & \forall x \forall y(E(x,y) \Rightarrow (\neg(R(x) \wedge R(y)) \wedge \\ & \qquad \neg(B(x) \wedge B(y)) \wedge \\ & \qquad \neg(G(x) \wedge G(y))))\end{aligned}$$

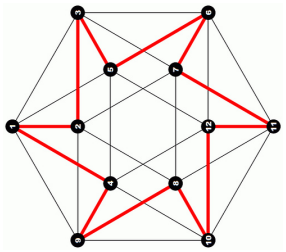
Second-Order Logic - Examples

- **Hamiltonian graph**: The following formula can test if a graph contains a Hamiltonian cycle.

$\exists L \exists S$ linear order$(L) \wedge$
S is the successor relation of $L \wedge$
$\forall x \exists y(L(x,y) \vee L(y,x)) \wedge$
$\forall x \forall y(S(x,y) \Rightarrow E(x,y))$

L is a linear ordering relation.
$(\forall x \neg L(x,x)) \wedge$
$(\forall x \forall y \forall z(L(x,y) \wedge L(y,z) \Rightarrow L(x,z))) \wedge$
$\forall x \forall y((x \neq y) \Rightarrow (L(x,y) \vee L(y,x)))$

S is a circular successor relation:
$\forall x \forall y S(x,y) \Leftrightarrow$
$((L(x,y) \wedge \neg \exists z(L(x,z) \wedge L(z,y))) \wedge$
$(\neg \exists z L(x,z) \wedge \neg \exists z L(z,y)))$



Second-Order Logic - Examples

- **Evenness**: The following formula can test if the size of a domain is even.

$$\begin{aligned}\exists A \exists B \quad & \forall x \exists y A(x,y) \wedge \\ & \forall x \forall y \forall z(A(x,y) \wedge A(x,z) \Rightarrow y = z) \wedge \\ & \forall x \forall y \forall z(A(x,z) \wedge A(y,z) \Rightarrow x = y) \wedge \\ & \forall x \forall y B(x) \wedge A(x,y) \Rightarrow \neg B(y) \wedge \\ & \forall x \forall y \neg B(x) \wedge A(x,y) \Rightarrow B(y)\end{aligned}$$

Characterizing NP

- **Theorem** (Fagin 1974)

  Let Q be a query over the class of **finite structures**. Then the following are equivalent:
  - Q is in NP.
  - Q is definable by ∃SO.

  $$\exists\mathbf{SO} \equiv \mathbf{NP}$$

- It reflects a normal form for NP algorithms:
  1. "**Guessing**" phase
     non-deterministic guess of a polynomially size bounded "certificate";
  2. "**Verifying**" phase
     deterministic polynomial time verification of this certificate.

Characterizing NP

- **Lemma**

  Every ∃SO definable query can be evaluated in NP.

- **Proof**:
  - Let Q be $\exists X_1 \ldots \exists X_n \varphi(X_1, \ldots, X_n)$, where φ is a FO formula.
  - Given $\mathfrak{A}$, the non-deterministic machine first

    Step 1 : guesses $R_1, \ldots, R_n$ for $X_1, \ldots, X_n$, and
    Step 2 : checks if $\varphi(R_1, \ldots, R_n)$ holds.
  - Step 2 can be done in polynomial time in the length of the encoding string of $\mathfrak{A}$ plus the size of $R_1, \ldots, R_n$.
  - Hence, the computation is polynomially time bounded.

Characterizing NP

- **Lemma**

  Every query over finite structures in NP can be expressed in ∃SO.

- **Proof**:
  - Suppose $M = (S, \Sigma, \Delta, \delta, q_0, S_a, S_r)$ is a nondeterministic polynomial time TM with a one-way infinite tape, where $\Sigma = \{0, 1\}$, $S = \{q_0, \ldots, q_{m-1}\}$ and $\Delta = \Sigma \cup \{-\}$.
  - The sentence describing acceptance by M on encodings of structures has the form

    $$\exists L \exists T_0 \exists T_1 \exists T_2 \exists H_{q_0} \ldots \exists H_{q_{m-1}} \varphi,$$

    where
    - L is linear order on the universe;
    - T_0, T_1, T_2 are tape predicates;
    - $\{H_q | q \in S\}$ are head predicates;
    - φ is a FO formula stating that when M starts on the encoding of $\mathfrak{A}$, the predicates T_i's and H_q's correspond to its computation, and eventually M reaches an accepting state.

Characterizing NP

- The class coNP consists of the problem whose complements are in NP, while the negation of an ∃SO sentence is an ∀SO sentence.

- **Corollary**

  ∀SO captures coNP.

- To show that NP $\neq$ coNP, it would suffice to exhibit a query definable in ∀SO but not definable in ∃SO. Hence, we have:

  $$\forall\mathbf{SO} \neq \exists\mathbf{SO} \Rightarrow \mathbf{NP} \neq \mathbf{coNP}$$
  $$\Rightarrow \mathbf{Ptime} \neq \mathbf{NP}$$

- Note that, the separation of ∃SO and ∀SO is specific to finite structures. Over some infinite structures (e.g., $\langle \mathbb{N}, +, \cdot \rangle$), the logics ∃SO and ∀SO are known to be different.

Fixed Point Logics

- Extending FO with fixed point operators to formalize inductive definitions.
  - FP: define new relations inductively;
  - SO: quantify over relations arbitrarily.
- **Transitive closure**: Given a binary relation E, the **transitive closure** of E is the smallest relation R satisfying:
  - $E \subseteq R$, and
  - if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.
- We can have a sequence of relations in the computation:

$$R^0, R^1, \ldots, R^n,$$

  where R^n is a fixed point of an operator that sends each R^i to R^{i+1}.
- Transitive closure is not expressible in FO.

Fixed Point Logics

- Let D be a finite set, and $\mathcal{P}(D)$ be its powerset. An **operator** on D is a mapping $F : \mathcal{P}(D) \mapsto \mathcal{P}(D)$.

  Some properties of operators:

  1. F is **monotone** if $X \subseteq Y$ implies $F(X) \subseteq F(Y)$;
  2. F is **inflationary** if $X \subseteq F(X)$ for all $X \in \mathcal{P}(X)$, i.e., the sequence $(F^n(X))_{n\in\mathbb{N}}$ is increasing in the sense that $X \subseteq F(X) \subseteq F(F(X)) \subseteq \ldots$

  (F **monotone**) $\Rightarrow$ (F **inflationary**)
- A set $X \subseteq D$ is a **fixed point** of F if $F(X) = X$.
- A set $X \subseteq D$ is a **least fixed point** of F, denoted as $\mathbf{lfp}(F)$, if it is a fixed point and $X \subseteq Y$ for every other fixed point Y of F.

Fixed Point Logics

- Every monotone operator has a least fixed point, which is the union of relations in an increasing sequence $X^0, X^1, \ldots, X^{i+1}$, i.e.,

$$\mathbf{lfp}(F) = \bigcup_{i=0}^{\infty} X^i,$$

- However, not all the operators are monotone.
  1. **Inflationary fixed point**

$$\mathbf{ifp}(F) = \bigcup_{i=0}^{n} X^i \cup F(X^i)$$

  2. **Partial fixed point**

$$\mathbf{pfp}(F) = \begin{cases} X^n & \text{if } X^n = X^{n+1} \\ \emptyset & \text{if } X^n \neq X^{n+1} \text{ for all } n \leq 2^{|D|}. \end{cases}$$

- If F is monotone, then $\mathbf{lfp}(F) = \mathbf{ifp}(F) = \mathbf{pfp}(F)$.

Fixed Point Logics - IFP and PFP

- We now add formulas for computing fixed points into FO.
- Suppose $\varphi(X, \vec{x})$ be a formula of $\sigma \cup \{X\}$, i.e., contains a SO variable X of arity k, and a tuple $\vec{x}$ of k free FO variables. For each structure $\mathfrak{A}$, φ defines an operator

$$F_\varphi : \mathcal{P}(A^k) \to \mathcal{P}(A^k)$$

  such that

$$F_\varphi(X) = \{\vec{a} | \mathfrak{A} \models \varphi(X/R, \vec{a})\}.$$

- **Syntax**: If $\varphi(X, \vec{x})$ is a formula, then $[fp_{X,\vec{x}}\varphi(X, \vec{x})](\vec{t})$ is also a formula, whose free variables are those of $\vec{t}$.
- **Semantics**:
  1. For IFP: $\mathfrak{A} \models [fp_{X,\vec{x}}\varphi(X, \vec{x})](\vec{a})$ iff $\vec{a} \in \mathbf{ifp}(F_\varphi)$.
  2. For PFP: $\mathfrak{A} \models [fp_{X,\vec{x}}\varphi(X, \vec{x})](\vec{a})$ iff $\vec{a} \in \mathbf{pfp}(F_\varphi)$.
- *Can we define an extension of FO with the least fixed point in exactly the same way?*

Fixed Point Logics - LFP

- Unfortunately, testing if F_φ is monotone is undecidable for FO formulas φ.
- To ensure that lfps are only taken for monotone operators, we impose syntactic restrictions.
  1. $\varphi(X, \bar{x})$ is **positive** in. X if every occurrence of X is inside of an even number of negations.

     Example: Are the following formulas positive in R or S?
     - $\exists x \neg R(x) \vee \neg \forall y \forall z \neg (R(y) \wedge \neg R(z))$
     - $R(x, y) \wedge \neg \exists y \forall z (\neg R(x, y) \wedge S(x, y))$
  2. If $\varphi(X, \bar{x})$ is positive in X, then F_φ is monotone (results from classical FO logic).
- **Syntax**: the same as IFP and PFP.
- **Semantics**:$\mathfrak{A} \models [fp_{X,\vec{x}}\varphi(X, \vec{x})](\vec{a})$ iff $\vec{a} \in \mathbf{lfp}(F_\varphi)$.

Fixed Point Logics - Examples

- Consider a graph whose edge relation is E.
  - Suppose that $\varphi(R, x, y)$ is defined by:

    $$E(x, y) \vee \exists z(E(x, z) \wedge R(z, y))$$

    1. **Transitive closure**:

       $$[\mathbf{lfp}_{R,x,y}\varphi(R, x, y)](u, v)$$

    2. **Graph connectivity**:

       $$\forall u \forall v [\mathbf{lfp}_{R,x,y}\varphi(R, x, y)](u, v)$$
  - Suppose that $\psi(R, x)$ is defined by:

    $$\forall y(E(y, x) \Rightarrow R(y))$$

    1. **Acyclicity**:

       $$\forall u [\mathbf{lfp}_{R,x}\psi(R, x)](u)$$

Fixed Point Logics - Examples

- Consider a graph whose edge relation is E, and $\varphi_1(R, x, y)$ defined by:

  $$E(x, y) \vee \exists z(E(x, z) \wedge R(z, y))$$

  Are the least, inflationary and partial fixed points of φ_1 the same?
- They all compute the **transitive closure** of the graph. In each case, $R^n = \{(u, v) | \text{there is a path between } u \text{ and } v \text{ of length less than } n\}$.

  $$[\mathbf{lfp}_{R,x,y}\varphi_1(R, x, y)](u, v) =$$
  $$[\mathbf{ifp}_{R,x,y}\varphi_1(R, x, y)](u, v) =$$
  $$[\mathbf{pfp}_{R,x,y}\varphi_1(R, x, y)](u, v)$$

Fixed Point Logics - Examples

- Consider a graph whose edge relation is E, and $\varphi_2(R, x, y)$ defined by:

  $$\exists z(E(x, z) \wedge R(z, y))$$

  Are the least, inflationary and partial fixed points of φ_2 the same?
- LFP and PFP return an empty set, i.e., $R^n = \emptyset$.

  $$[\mathbf{lfp}_{R,x,y}\varphi_1(R, x, y)](u, v) =$$
  $$[\mathbf{pfp}_{R,x,y}\varphi_1(R, x, y)](u, v)$$
- IFP still computes the **transitive closure** of the graph, i.e, $R^n = \{(u, v) | \text{there is a path between } u \text{ and } v \text{ of length less than } n\}$.

IFP vs LFP

- **Theorem** (Gurevich and Shelah, 1986)
  For every formula in IFP there is an LFP formula equivalent to it on all finite structures, i.e.,

  $$\text{LFP} \equiv \text{IFP over all finite structures}$$

- **Theorem** (Kreutzer, 2002)
  For every formula in IFP there is an equivalent formula in LFP, i.e.,

  $$\text{LFP} \equiv \text{IFP over all structures}$$

- These results give us freedom to formalize inductive definitions in IFP (rather than in LFP) – don't need to check whether or not a formula is positive!

IFP vs LFP vs PFP

- Expressiveness of fixed point logics over all structures:

  $$\text{FO} \subsetneq \text{LFP} \equiv \text{IFP} \subseteq \text{PFP}$$

- Adding an order can increase their expressiveness, i.e.,

  $$\text{LFP} \subsetneq \text{LFP} + <$$
  $$\text{IFP} \subsetneq \text{IFP} + <$$
  $$\text{PFP} \subsetneq \text{PFP} + <$$

- The query that separates these logics on ordered and unordered structures is **evenness**.
  - **evenness** is not expressible in any of LFP, IFP and PFP.
  - **evenness** is expressible in LFP + $<$, IFP + $<$ and PFP + $<$.

Characterizing Ptime

- **Theorem** (Immerman and Vardi, 1982)
  Let Q be a query over the class of **ordered finite structures**. Then the following are equivalent:
  - Q is in Ptime.
  - Q is definable by LFP.

  $$\text{LFP} + < \equiv \text{IFP} + < \equiv \text{Ptime}$$

- The **restriction to ordered structures** is:
  1. **necessary** for enabling sufficient coding machinery;
     (algorithms/Turing machines always work with an ordered input representation)
  2. **unsatisfactory** for guarantee semantic independence.
     (the answer of a query needs to be independent of the ordering in an input representation, but there are $n!$ many orderings to consider!)

Characterizing Ptime

- **Lemma**
  Every LFP definable query can be evaluated in Ptime over ordered finite structures.
- **Proof**:
  - By induction on formulas
    - Formulas with only Boolean connectives and quantifiers can be handled as for FO.
    - For formulas of the form $[\mathbf{ifp}_{X,\vec{x}}\varphi](\vec{t})$, $F_\varphi : \mathcal{P}(A^k) \to \mathcal{P}(A^k)$ is a Ptime-computable monotone operator (i.e., to compute X^{i+1} from X^i, it goes through all $\vec{a} \in A^k$ and evaluates φ).
  - Since the fixed point computation stops after at most $|A^k|$ iteration, and in each iteration $[\mathbf{ifp}_{X,\vec{x}}\varphi](\vec{t})$ can be computed in polynomial time, every LFP formula can be evaluated in Ptime.

Characterizing Ptime

- **Lemma**

  Every Ptime query of ordered finite structures can be expressed in LFP.

- **Proof**:
  - Suppose that $M = (S, \Sigma, \Delta, \delta, q_0, S_a, S_r)$ is a deterministic polynomial time TM with a one-way infinite tape, where $\Sigma = \{0, 1\}$ and $\Delta = \Sigma \cup \{-\}$. The linear order that is defined existentially in the proof of the only if direction of

Characterizing Pspace

- **Theorem** (Abiteboul and Vianu, 1982)

  Let Q be a query over the class of **ordered finite structures**. Then the following are equivalent:
  - Q is in Pspace.
  - Q is definable by PFP.

  PFP + $<$ $\equiv$ Pspace

- **Lemma**

  every Pspace query of ordered finite structures can be expressed in PFP

  (The proof can be done by modifying the proof of the Immerman-Vardi theorem to simulate the accepting condition of a Turing machine using a PFP formula.)

Characterizing Pspace

- **Lemma**

  Every PFP definable query can be evaluated in Pspace over ordered finite structures.

- **Proof**:
  - We need to show that the evaluation of F_φ for a partial fixed point $[\mathbf{pfp}_{X,\vec{x}}\varphi](\vec{t})$ with k-ary X is in Pspace.
  - By the definition, one only need to check the following:

$$\mathbf{pfp}(F) = \begin{cases} X^n & \text{if } X^n = X^{n+1} \\ \emptyset & \text{if } X^n \neq X^{n+1} \text{ for all } n \leq 2^{|D|}. \end{cases}$$

    For that, it suffices to enumerate all the subsets of A^k, one by one (which can be done in Pspace).
  - Since only $2^{|D|^k}$ steps need to be made, the entire computation is in Pspace.

Ptime vs Pspace

- Whether *Ptime* $\subsetneq$ *Pspace* is one of the open problems in computational complexity theory.
- Based on their logical definability, we have:

  Ptime $\equiv$ Pspace $\Leftrightarrow$ LFP $\equiv$ PFP over all ordered finite structures

- **Theorem** (Abiteboul and Vianu, 1991)

  The following are equivalent:
  - LFP and PFP have the expressive power over all finite structures.
  - Pspace collapses to Ptime.

  Ptime $\equiv$ Pspace $\Leftrightarrow$ LFP $\equiv$ PFP over all finite structures

  That is, the restriction to ordered structures can indeed be removed.

The Role of Order

- No results show that a logic can characterize a complexity class below NP on the class of all finite structures.
- Order provides a specific encoding, but algorithms in a complexity class should yield the same result on all different encodings of the input.

Open Problem – A Logic for P?

- **Open problem** (Chandra and Harel, 1982)

  Is there a logic that captures Ptime on all and not just ordered structures?
- Recall that, **LFP +** $\leq \equiv$ **Ptime**.
- *Can we drop the ordering* $\leq$ *but still capture Ptime*?
  - If dropping "ordered" from Immerman and Vardi's Theorem, then the theorem would be false.
  - **evenness** is not expressible in LFP, IFP or PFP.

Open Problem – A Logic for P?

- **Conjecture** (Gurevich, 1988)

  There is no logic that captures Ptime on the class of all finite structures.
- *Can we enrich LFP so that LFP can express counting properties? Can we restrict structures so that the ordering is not necessary?*
  - Study various extensions of FO, such as generalizing qualifiers and adding fixed-point operators
  - Study various restrictions on finite structures, such as the class of all graphs with certain properties.

Open Problem – A Logic for P?

- Let L be a logic and $\mathfrak{S}$ a class of finite structures. Then L **captures Ptime on** $\mathfrak{S}$ if it satisfies the following conditions:
  1. The set of sentences in L is decidable.
  2. There is an algorithm that associates with every sentence φ in L a polynomial time algorithm that decides P_φ on $\mathfrak{S}$.
  3. For every polynomial time algorithm that decides a query P on $\mathfrak{S}$, there is a sentence φ in L that defines P on $\mathfrak{S}$.
- L **captures Ptime** if L captures Ptime on the class of **all finite structures**.

Open Problem – A Logic for P?

- Let L be a logic and $\mathfrak{S}$ a class of finite structures. Then *L captures Ptime on $\mathfrak{S}$* if it satisfies the following conditions:
  1. The set of sentences in L is decidable.
  2. There is an algorithm that associates with every sentence φ in L a polynomial time algorithm that decides P_φ on $\mathfrak{S}$.
  3. For every polynomial time algorithm that decides a query P on $\mathfrak{S}$, there is a sentence φ in L that defines P on $\mathfrak{S}$.
- *L captures Ptime* if L captures Ptime on the class of **all finite structures**.
- FO and IFP only satisfy (1) and (2) for the class of all finite structures, but does not satisfy (3).

Open Problem – A Logic for P?

- Let L be a logic and $\mathfrak{S}$ a class of finite structures. Then *L captures Ptime on $\mathfrak{S}$* if it satisfies the following conditions:
  1. The set of sentences in L is decidable.
  2. There is an algorithm that associates with every sentence φ in L a polynomial time algorithm that decides P_φ on $\mathfrak{S}$.
  3. For every polynomial time algorithm that decides a query P on $\mathfrak{S}$, there is a sentence φ in L that defines P on $\mathfrak{S}$.
- *L captures Ptime* if L captures Ptime on the class of **all finite structures**.
- Without (2), we could cheat ... Let $\mathfrak{P}$ be polynomial time decidable queries over all finite structures. Then we could have a "logic":
  - $L := \{\varphi | P_\varphi \in \mathfrak{P}\}$;
  - $\mathfrak{A} \models \varphi$ if $\mathfrak{A} \in P_\varphi$ for every structure $\mathfrak{A}$ and $\varphi \in L$.

Open Problem – A Logic for P?

- *A candidate logic*: IFP + C (the extension of FO with inflationary fixed-points and counting quantifiers).
- **Some negative results**
  - **Theorem** (Cai et al., 1992)
    IFP + C does not express all polynomial time properties of graphs.
- **Some positive results**

  IFP + C does capture Ptime on specific classes of finite structures (the key idea is to find an IFP+C-definable linear order on restricted classes).
  - **Theorem** (Grohe, 1998)
    IFP + C captures Ptime on all planar graphs.
  - **Theorem** (Grohe and J. Mariño, 1999)
    IFP + C captures Ptime on all graphs with bounded tree-width.
  - **Theorem** (Grohe, 2010))
    IFP + C captures Ptime on every class of graphs that exclude a minor.